MATH 207(c)–Analysis IV

1. For the following functions, determine the nature of the singularity at $z = z_0$ (i.e. regular point, pole or essential singularity), compute the residue, calculate the radius of convergence of the Laurent series.

(a) $f(z) = \frac{z^2+3z+2}{z+1}$, $z_0 = -1$,

(b) $f(z) = \frac{(z+1)^{\frac{1}{3}}}{z}$, $z_0 = 0$,

(c) $f(z) = e^{\frac{z^2+1}{z-i}}$, $z_0 = i$,

(d) $f(z) = \frac{z^{-7}+1}{1+z}$, $z_0 = 0$.

2. Let γ be a simple, closed curve in $\mathbb{C}$ which is counterclockwise oriented. What are the possible values of the following integrals, depending on the shape of γ?

(a) $\int_\gamma \frac{1}{z(z+2)}\,dz$,

(b) $\int_\gamma e^{\frac{1}{z^2}}\,dz$,

(c) $\int_\gamma \frac{e^{iz}}{z^4+1}\,dz$,

(d) $\int_\gamma \frac{\sin(z)}{z}$.

3. Let γ the circle of radius 2 centered at the origin, parametrized counter-clockwise. What is the value of the integral

$$\int_\gamma \tan(z)\,dz,$$

where, as usual, $\tan(z) = \frac{\sin(z)}{\cos(z)}$.

4. Let $\mathcal{U} \subseteq \mathbb{C}$ be an open set and $p, q : \mathcal{U} \to \mathbb{C}$ be holomorphic functions and consider the function $f(z) = \frac{p(z)}{q(z)}$ defined at the points where $q(z) \neq 0$. Let also z_0 be a point in $\mathcal{U}$ such that $q(z_0) = 0$ (i.e. a singularity of f).

(a) Assume that $p(z_0) \neq 0$ and that q vanishes to first order at z_0, i.e. $q(z_0) = 0$ but $q'(z_0) \neq 0$. Show that $\mathrm{Res}_{z_0}(f) = \dfrac{p(z_0)}{q'(z_0)}$.

(b) Assume that p vanishes to first order at z_0 and that q vanishes to second order at z_0, i.e. $q(z_0) = q'(z_0) = 0$ but $q''(z_0) \neq 0$. Show that $\mathrm{Res}_{z_0}(f) = \dfrac{2p'(z_0)}{q''(z_0)}$.

5. Compute the following integral:

$$\int_0^{2\pi} \frac{\cos^2(\theta)}{13 - 5\cos(2\theta)}\, d\theta.$$

Hint: Use the residue theorem, by recasting the above as a complex integral over the unit circle. For $z = e^{i\theta}$, you might need to use the identity

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\Big(z + \frac{1}{z}\Big)$$

(and similarly for $\cos(2\theta)$).

Solutions

1. **(a)** $f(z) = \frac{z^2+3z+2}{z+1} = \frac{(z+1)(z+2)}{(z+1)} = z+2$ This function has a removable singularity in $z=-1$ since on can extend it by continuity at this point. The residue is thus zero and the radius of convergence is infinite (since there exists no other singularity).

(b) The singularity at $z=0$ is directly given as a pole of order one, since

$$\lim_{z\to 0} zf(z) = \lim_{z\to 0} z \cdot \frac{(1+z)^{1/3}}{z} = 1 \neq 0.$$

This limit is also the residue of f (it is the formula for the case of a simple pole). The radius of convergence is $R=1$ (since the domain of holomorphicity for $(z+1)^{\frac{1}{3}} = e^{\frac{1}{3}\log(1+z)}$ is $\mathbb{C}\setminus(-\infty,-1]$).

(c) $f(z) = e^{\frac{z^2+1}{z-i}} = e^{z+i}$ is also a function with a removable singularity in $z=i$. thus, the residue is zero. The convergence radius is infinite since there is no other singularity.

(d) One can construct the Laurent series as:

$$f(z) = \frac{z^{-7}+1}{1+z} = \frac{1}{z^7}\cdot\frac{1+z^7}{1+z} = \frac{1}{z^7}\cdot\frac{(1+z)(1-z+z^2-z^3+z^4-z^5+z^6)}{1+z} = \frac{1}{z^7}(1-z+z^2-z^3+z^4-z^5$$

The residue is the coefficient of z^{-1}, so $\mathrm{Res}_0(f) = 1$.

2. In this exercise, we consider $\gamma \subset \mathbb{C}$ to be a simply connected, closed, and positively oriented (i.e. counter-clockwise) curve. The different cases to be considered come down to count how many poles are inside the domain defined by $\mathrm{Int}(\gamma)$, or if any pole belongs to the curve γ, in which case the integral is not well defined.

(a) The poles of the function $f(z) = \frac{1}{z(z+2)}$ are $z=0$ and $z=-2$. Both poles are of order 1, thus we can easily compute their residue using the formula (for the case of simple poles) $Res_{z_0}(f) = \lim_{z\to z_0}(z-z_0)f(z)$: So $\mathrm{Res}(z=0) = 1/2$ and $\mathrm{Res}(z=-2) = -1/2$. We can distinguish the following cases:

$$\int_\gamma f(z)\,dz = \begin{cases} 0\,, & \{0,-2\} \not\subset \mathrm{Int}(\gamma) \text{ or } \{0,-2\} \subset \mathrm{Int}(\gamma) \\ \frac{1}{2}\,, & 0 \in \mathrm{Int}(\gamma) \text{ and } -2 \notin \mathrm{Int}(\gamma) \\ -\frac{1}{2}\,, & -2 \in \mathrm{Int}(\gamma) \text{ and } 0 \notin \mathrm{Int}(\gamma) \\ \text{ill defined}\,, & 0 \in \gamma \text{ or } -2 \in \gamma. \end{cases}$$

(b) The function $f(z) = e^{1/z^2}$ has an essential singularity at $z=0$ since its Laurent series exhibits singular part with an infinite number of terms:

$$e^{\frac{1}{z^2}} = \sum_{n=0}^{\infty} \frac{z^{-2n}}{n!} = 1 + \frac{1}{z^2} + \frac{1}{2z^4} + \frac{1}{6z^6} + \ldots$$

The coefficient of the term z^{-1} is zero, so the residue at this point is also zero, by definition. The integral is null, regardless of whether $z=0$ is inside or outside $\mathrm{Int}(\gamma)$, and is ill defined if $0 \in \gamma$.

(c) The function $f(z) = \frac{e^{iz}}{z^4+1}$ has four poles of degree one each on the unitary circle. These poles are of the form $z_k = e^{i(\pi/4+k\pi/2)}$ with $k \in \{0,1,2,3\}$ such that one can decompose $z^4+1 = (z-z_0)(z-z_1)(z-z_2)(z-z_3)$. We explicit the computation of the residue at $z = z_0$:

$$\begin{aligned}\text{Res}(z=z_0) &= \lim_{z\to z_0}(z-z_0)\cdot\frac{e^{iz}}{(z-z_0)(z-z_1)(z-z_2)(z-z_3)} \\ &= \frac{e^{iz_0}}{(z_0-z_1)(z_0-z_2)(z_0-z_3)} = \frac{e^{-\sqrt{2}/2}e^{i\sqrt{2}/2}}{\sqrt{2}\cdot(\sqrt{2}+i\sqrt{2})\cdot i\sqrt{2}} \\ &= -\frac{e^{-\sqrt{2}/2}e^{i\sqrt{2}/2}}{4}\left(\frac{1+i}{\sqrt{2}}\right) = -\frac{e^{-\sqrt{2}/2}}{4}e^{i\left(\frac{\sqrt{2}}{2}+\frac{\pi}{4}\right)}\end{aligned}$$

Similarly, we compute the other residues at $z_{k1,2,3}$:

- $\text{Res}(z=z_1) = \frac{e^{-\sqrt{2}/2}}{4}e^{-i\left(\frac{\sqrt{2}}{2}+\frac{\pi}{4}\right)}$
- $\text{Res}(z=z_2) = \frac{e^{\sqrt{2}/2}}{4}e^{i\left(-\frac{\sqrt{2}}{2}+\frac{\pi}{4}\right)}$
- $\text{Res}(z=z_3) = -\frac{e^{\sqrt{2}/2}}{4}e^{i\left(\frac{\sqrt{2}}{2}-\frac{\pi}{4}\right)}$

The curve γ can enclose all the combinations of either of these four poles. We emphasize here four kinds of these combinations (cf. Figure below):

- Sum of the two residues in the upper plane. We define $\theta = \frac{\sqrt{2}}{2}+\frac{\pi}{4}$ and $A = \frac{e^{-\sqrt{2}/2}}{4}$;

$$\text{Res}(z_0)+\text{Res}(z_1) = -Ae^{i\theta}+Ae^{-i\theta} = -A\left(e^{i\theta}-e^{-i\theta}\right) = -A2i\sin(\theta)$$

- Sum of two residues on the diagonal. We define $\tilde{z} = e^{i\pi/4}$;

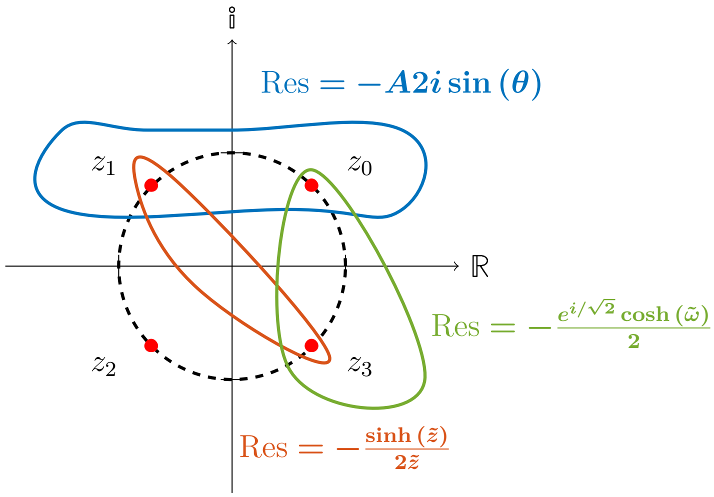
$$\begin{aligned}\text{Res}(z_1)+\text{Res}(z_3) &= \frac{e^{-\sqrt{2}/2}}{4}e^{-i\left(\frac{\sqrt{2}}{2}+\frac{\pi}{4}\right)} - \frac{e^{\sqrt{2}/2}}{4}e^{i\left(\frac{\sqrt{2}}{2}-\frac{\pi}{4}\right)} \\ &= -\frac{e^{-i\pi/4}}{4}\left(e^{\frac{1+i}{\sqrt{2}}}-e^{-\frac{1+i}{\sqrt{2}}}\right) = -\frac{e^{-i\pi/4}\sinh\left(e^{i\pi/4}\right)}{2} = -\frac{\sinh(\tilde{z})}{2\tilde{z}}\end{aligned}$$

- Sum of two residues with the same real component. We define $\tilde{\omega} = \frac{\sqrt{2}}{2} - i\frac{\pi}{4}$;

$$\begin{aligned}\text{Res}(z_0)+\text{Res}(z_3) &= -\frac{e^{-1/\sqrt{2}}}{4}e^{i\left(\frac{1}{\sqrt{2}}+\frac{\pi}{4}\right)} - \frac{e^{1/\sqrt{2}}}{4}e^{i\left(\frac{1}{\sqrt{2}}-\frac{\pi}{4}\right)} \\ &= -\frac{e^{i/\sqrt{2}}}{4}\left(e^{-\frac{1}{\sqrt{2}}+i\frac{\pi}{4}}+e^{\frac{1}{\sqrt{2}}-i\frac{\pi}{4}}\right) = -\frac{e^{i/\sqrt{2}}\cosh(\tilde{\omega})}{2}\end{aligned}$$

- Sum of all the residues $\sum_i \text{Res}(z_{ki}) = 0$

Finally, the integral is ill defined if the curve if one or several of these poles belongs to the curve γ.



Summation sketch of the residues.

(d) The function $f(z) = \frac{\sin(z)}{z}$ has a removable singularity in $z = 0$, thus this integral is always null, whether if $z = 0$ belongs or not to $\text{Int}(\gamma)$, and even if it belongs to the curve γ itself.

3. As already stated in exercise 2(b) of the exercise sheet 4, the complex function $\cos(z)$ admits the same zeros as its real counterpart given by $z_k = \pi/2 + k\pi$ with $k \in \mathbb{Q}$. With a circular curve γ, centered at the origin and with a radius $r = 2$, there are two poles of $\tan(z)$ that are contained inside $\text{Int}(\gamma)$, namely $z_\pm = \pm\pi/2$. By developing the cosine Laurent series around these points, one can compute the residues:

$$\text{Res}\left(z = \frac{\pi}{2}\right) = \lim_{z\to\pi/2}\left(z - \frac{\pi}{2}\right) \cdot \frac{\sin(z)}{-\left(z - \frac{\pi}{2}\right) + \frac{1}{6}\left(z - \frac{\pi}{2}\right)^3 - \mathcal{O}\left(\left(z - \frac{\pi}{2}\right)^5\right)}$$

$$= \lim_{z\to\pi/2} \frac{\sin(z)}{-1 + \frac{1}{6}\left(z - \frac{\pi}{2}\right)^2 - \mathcal{O}\left(\left(z - \frac{\pi}{2}\right)^4\right)} = -1$$

Similarly, one finds $\text{Res}\left(z = -\frac{\pi}{2}\right) = -1$, such that $\int_\gamma \tan(z)\,dz = -4\pi i$.

4. In this exercise, we use the property that a function *vanishes at the n^{th} order* to construct its Taylor series up to the order $n + 1$.

(a) Since $p(z_0) \neq 0$, we can write:

$$\text{Res}_{z_0}(f) = \lim_{z \to z_0} (z - z_0) \cdot \frac{p(z)}{\cancelto{0}{q(z_0)} + q'(z_0)(z - z_0) + \frac{q''(z_0)}{2}(z - z_0)^2 + \dots}$$

$$= \lim_{z \to z_0} \frac{p(z)}{q'(z_0) + \frac{q''(z_0)}{2}(z - z_0) + \dots} = \frac{p(z_0)}{q'(z_0)} \neq 0$$

which indicates a pole of order one since this limit is not null, and gives the value of the residue by definition.

(b) Similarly, we write:

$$\text{Res}_{z_0}(f) = \lim_{z \to z_0} (z - z_0) \cdot \frac{\cancelto{0}{p(z_0)} + p'(z_0)(z - z_0) + \frac{p''(z_0)}{2}(z - z_0)^2 + \dots}{\cancelto{0}{q(z_0)} + \cancelto{0}{q'(z_0)}(z - z_0) + \frac{q''(z_0)}{2}(z - z_0)^2 + \frac{q'''(z_0)}{3!}(z - z_0)^3 + \dots}$$

$$= \lim_{z \to z_0} \frac{p'(z_0) + \frac{p''(z_0)}{2}(z - z_0) + \dots}{\frac{q''(z_0)}{2} + \frac{q'''(z_0)}{3!}(z - z_0) + \dots} = \frac{2p'(z_0)}{q''(z_0)} \neq 0$$

following the same reasoning as above, this is the value of the residue.

5. We use θ as the parameter that describes the unitary circle $\gamma(\theta) = e^{i\theta}$ with $\theta \in [0, 2\pi]$. This leads to the following change of variable: $\{z \to e^{i\theta}\ ;\ dz \to ie^{i\theta} d\theta\}$. Note that $\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + \frac{1}{z}}{2}$ and $\cos(2\theta) = \frac{e^{2i\theta} + e^{-2i\theta}}{2} = \frac{z^2 + \frac{1}{z^2}}{2}$ We can then write:

$$\int_0^{2\pi} \frac{\cos^2(\theta)}{13 - 5\cos(2\theta)} \, d\theta = \int_0^{2\pi} \frac{\cos^2(\theta)}{13 - 5\cos(2\theta)} \, \frac{1}{ie^{i\theta}} ie^{i\theta} d\theta$$

$$= \int_\gamma \frac{\frac{1}{4}\left(z + \frac{1}{z}\right)^2}{13 - \frac{5}{2}\left(z^2 + \frac{1}{z^2}\right)} \left(\frac{-i}{z}\right) dz = \int_\gamma \frac{i(z^4 + 2z^2 + 1)}{10z^5 - 52z^3 + 10z} \, dz$$

$$= \int_\gamma \frac{i(z^2 + 1)^2}{2z(z^2 - 5)(5z^2 - 1)} \, dz$$

$$= \int_\gamma \frac{i(z^2 + 1)^2}{2z\left(z + \sqrt{5}\right)\left(z - \sqrt{5}\right)\left(\sqrt{5}z + 1\right)\left(\sqrt{5}z - 1\right)} \, dz$$

$$= \int_\gamma f(z) \, dz$$

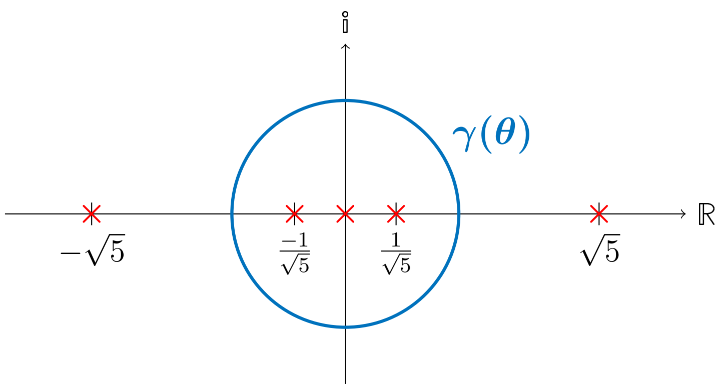
There are 5 poles of order one, among which 3 belongs to the interior of γ: $z \in \{0, \pm 1/\sqrt{5}\}$ as one can see in the figure below. We compute their respective residue:

$$\text{Res}(z=0) = \lim_{z\to 0} \; z \cdot f(z) = \frac{i}{10}$$
$$\text{Res}(z=\frac{1}{\sqrt{5}}) = \lim_{z\to \frac{1}{\sqrt{5}}} \; (z - \frac{1}{\sqrt{5}}) \cdot f(z) = -\frac{3i}{40}$$
$$\text{Res}(z=-\frac{1}{\sqrt{5}}) = \lim_{z\to -\frac{1}{\sqrt{5}}} \; (z + \frac{1}{\sqrt{5}}) \cdot f(z) = -\frac{3i}{40}$$

We conclude by the residue theorem:

$$\int_0^{2\pi} \frac{\cos^2(\theta)}{13 - 5\cos(2\theta)} \, d\theta = \int_\gamma f(z)\, dz = 2\pi i \sum_i \text{Res}(z_i) = 2\pi i \left(\frac{i}{10} - 2 \cdot \frac{3i}{40}\right) = \frac{\pi}{10}.$$

As a consistency check, we expect an answer in $\mathbb{R}$ since the integral is initially a real integral of a real function.



The poles of $f(z)$ are indicated by the red crosses.